Practice Problems for the Final Exam

1. For each of the following vectors $\mathbf{x}$, find the projection of $\mathbf{x}$ in the direction of $\mathbf{v}$.

(a) Find the projection of $\mathbf{x} = (3, 3, 3)$ in the direction of $\mathbf{v} = (2, 4, 6)$.

(b) Find the projection of $\mathbf{x} = (1, 2, 3)$ in the direction of $\mathbf{v} = (1, 1, 1)$.

(c) Find the projection of $\mathbf{x} = (1, 2)$ in the direction of $\mathbf{v} = (3, 4)$.

2. For each of the following lines, provide a **parametrization** of the line in the form $\mathbf{r}(t) = \mathbf{p_0} + t\mathbf{d}$ for $\mathbf{p_0}, \mathbf{d} \in \mathbb{R}^n$ for appropriate n.

(a) The line in $\mathbb{R}^3$ passing through $\mathbf{p} = (1, 2, 3)$ with direction vector $\mathbf{d} = (4, 5, 6)$.

(b) The line in $\mathbb{R}^2$ satisfying the equation $y = \dfrac{3}{2}x + 4$

(c) The line in $\mathbb{R}^3$ passing through the points $\mathbf{p} = (3, 2, 1)$ and $\mathbf{q} = (1, 2, 3)$.

3. For each of the following planes, determine the **implicit equation** of the plane in the form $ax+by+cz = d$ for scalars $a, b, c, d \in \mathbb{R}$.

(a) The plane passing through the points $\mathbf{p_1} = (3, 4, 5)$, $\mathbf{p_2} = (2, 3, 1)$, and $\mathbf{p_3} = (5, 5, 2)$.

(b) The plane with normal vector $\mathbf{N} = (1, 2, 3)$ passing through the point $\mathbf{p} = (3, 2, 1)$.

(c) The plane with parametrization $P(t, s) = (1, 2, 3) + (1, 2, 0)t + (0, 2, 1)s$ for $t, s \in \mathbb{R}$.

(d) The plane with direction vectors $\mathbf{d_1} = (1, 2, 2)$ and $\mathbf{d_2} = (2, 2, 1)$ intersecting the origin.

4. For each matrix, determine if the matrix is invertible. If it is invertible, find its inverse.

(a) $\mathbf{A_1} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$

(b) $\mathbf{A_2} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

(c) $\mathbf{A_3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

(d) $\mathbf{A_4} = \begin{pmatrix} 1 & 3 & -5 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$

(e) $\mathbf{A_5} = \begin{pmatrix} 2 & 0 & 4 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{pmatrix}$

(f) $\mathbf{A_6} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

5. Determine the solution set of the following systems, i.e. identify **all** solutions. If there are infinite solutions, express them as a linear combination of vectors with free variables as the scalars.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & -4 \\ 1 & -2 & 3 & -4 \\ 1 & 2 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 3 & 4 \\ 1 & 2 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$

6. Determine the **left multiplication matrices** for the following linear transformations.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $T(\mathbf{e_2}) = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, and $T(\mathbf{e_1}) = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_2 + 5x_3 \\ 6x_1 + 7x_3 \end{pmatrix}$

(c) The reflection $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ across the line $y = \frac{1}{2}x$.

(d) The rotation $T_4 : \mathbb{R}^2 \to \mathbb{R}^2$ by $\theta = \frac{5\pi}{6}$ counterclockwise.

(e) The orthogonal projection $T_5 : \mathbb{R}^2 \to \mathbb{R}^2$ by onto the line spanned by $\mathbf{v} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

(f) The reflection transformation $T_6 : \mathbb{R}^3 \to \mathbb{R}^3$ across the plane given by $x + 2x - 3z = 0$.

(g) The orthogonal projection $T_7 : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane with normal vector $\mathbf{N} = (1, -2, 3)^\top$ passing through the origin.

7. Determine if the following sets of vectors are linearly independent. If the set is linearly dependent, determine a linearly independent spanning set.

(a) $V_1 = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

(b) $V_2 = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}$

(c) $V_3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$

(d) $V_4 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} \right\}$

(e) $V_5 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \right\}$

(f) $V_6 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix} \right\}$

8. Find the determinant of the following matrices.

(a) $\mathbf{B_1} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ -2 & -8 & -10 \end{pmatrix}$

(b) $\mathbf{B_2} = \begin{pmatrix} 3 & 4 & -5 \\ 4 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix}$

(c) $\mathbf{B_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\mathbf{B_4} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

9. For each of the following matrices $\mathbf{M}$, determine the following:

1. the **characteristic polynomial** of $\mathbf{M}$,
2. the **real eigenvalues** of $\mathbf{M}$,
3. the **eigenvectors** corresponding to the real eigenvalues of $\mathbf{M}$

(a) $\mathbf{M_1} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$

(b) $\mathbf{M_2} = -\frac{1}{5}\begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix}$

(c) $\mathbf{M_3} = \begin{pmatrix} -20 & 11 \\ 4 & 0 \end{pmatrix}$

(d) $\mathbf{M_4} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(e) $\mathbf{M_5} = \begin{pmatrix} -2 & -1 & 8 \\ 12 & 6 & -16 \\ 6 & -1 & 0 \end{pmatrix}$

(f) $\mathbf{M_6} = \begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix}$

(g) $\mathbf{M_7} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$